A CONTRIBUTION TO THE SOLUTION OF MIXED PROBLEMS OF TRANSONIC AERODYNAMICS

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INTRODUCTION

I think I shall not be far from the truth in saying that most of the work in theoretical aerodynamics, except some rare "happy" cases of exact solutions, consists of simplifying general equations of aerodynamics in such a way that an analytical, or relatively easy numerical, solution of those simplified equations can be obtained.

The advent of electronic computers allowed us to set them the task of solving nonsimplified equations numerically, to a given accuracy in sufficiently general cases. In new branches of technique, such as for atomic physics and automatic control, electronic computers are widely used. I would say that in these branches computation is the basic method in investigations and design. Considering the rapid development of modern science, aerodynamics can be regarded as an old science. The methods of aerodynamical investigations had been established long before electronic computers appeared. They were mainly experimental methods (S.A.S. wind-tunnel experiments), which took only similarity laws from theoretical aerodynamics. Now, however, aerodynamics is undergoing revolutionary changes.

Till recently aerodynamics was a part of mechanics whose basic laws were established by Euler and Navier–Stokes. Modern aerodynamics encroaches upon the domains of chemical kinetics, kinetic theory of gases and electrodynamics. Differential equations of aerodynamics become much more complicated and I doubt whether it is possible to construct experimental equipment which will enable one to go directly from model experiments to actual testing, as it has been up to now. It seems very probable that in future aerodynamics, experiments will give only basic relations and constants while aerodynamic and thermal characteristics (missiles, sputniks, cosmic rockets) will be calculated.

In view of this I consider it very important to work out the methods of solution of non-simplified equations of aerodynamics for use on electronic computers. As the first step in this direction in the Computing

Center of the Academy of Sciences of the U.S.S.R. and some other institutes, the problem was set to develop the methods of solving the equations of classical aerodynamics using electronic computers. The most difficult problems here are mixed problems, where in some parts of flow velocities are subsonic (elliptic subdomain) and in others, supersonic (hyperbolic subdomain).

This report considers some results obtained in the solving of these problems.

DESCRIPTION OF THE METHOD

1. The partial differential equations of physics and mechanics that express certain laws of conservation (of mass, energy, charge, etc.) can be in many cases represented in a "divergence" form. With two independent variables the equation is written as follows

$$\frac{\partial P_i(x, y; u_1, u_2, \ldots u_n)}{\partial x} + \frac{\partial Q_i(x, y; u_1\, u_2, \ldots, u_n)}{\partial y} = F_i(x, y; u_1, u_2, \ldots, u_n) \quad (i = 1, 2, \ldots, n) \tag{1}$$

Here x, y are the independent variables, $u_1, \ldots, u_n$ the sought values, P_i, Q_i, F_i are the known functions of their arguments.

Now we consider the solution of system (1) in the rectangular region $a \leqslant x \leqslant b$, $c \leqslant y \leqslant d$ with the boundary conditions

$$\begin{aligned} &\text{at } x = a \quad \phi_\nu (y, \bar{u}_1, \ldots, \bar{u}_n) = 0 \quad (\nu = 1, 2, \ldots, k) \\ &\text{at } x = b \quad \phi_\nu (y, u_1, \ldots, u_n) = 0 \quad (\nu = k + 1, k + 2, \ldots, n) \end{aligned} \tag{2}$$

$$\begin{aligned} &\text{at } y = c \quad \psi_\nu (x, u_1, \ldots, u_n) = 0 \quad (\nu = 1, 2, \ldots, b) \\ &\text{at } y = d \quad \psi_\nu (x, u_1, \ldots, u_n) = 0 \quad (\nu = 1 + 1, 1 + \nu, \ldots, n) \end{aligned} \tag{3}$$

In a particular case the rectangular may degenerate into a semi-band ($b = +\infty$) or a band ($a = -\infty$, $b = +\infty$).

Now we divide the region into N bands with the lines parallel to the axis x ($y = y_k$, $k = 1, 2, \ldots, N - 1$). For the sake of simplicity we consider the bands to have equal width. Then we integrate each equation of system (1) across each band. As a result we obtain a system of integral relations

$$\frac{\mathrm{d}}{\mathrm{d}x} \int_{y_k}^{y_{k+1}} P_i \mathrm{d}y + Q_{i,k+1} - Q_{i,k} = \int_{y_k}^{y_{k+1}} F_i \, \mathrm{d}y \tag{4}$$

$$(i = 1, 2, \ldots, n; \qquad k = 0, 1, 2, \ldots, N - 1 \qquad y_0 = c\; y_N = \mathrm{d})$$

where $Q_{i,k}$ is the value of function Q_i at $y = y_k$. If now for the functions

P_i, F_i we apply any interpolation formula expressing the value of $P_i(F_i)$ at any y through its values on the lines $y = y_k$ and integrate it we shall obtain:

$$\left.\begin{aligned} \int_{y_k}^{y_{k+1}} P_i \, dy &= \delta \sum_{\alpha=0}^{d=N} A_{k,\alpha} P_{i,\alpha}, \qquad (\delta = d - c) \\ \int_{y_k}^{y_{k+1}} F_i \, dy &= \delta \sum_{\alpha=0}^{\alpha=N} A_{k,\alpha} F_{i,\alpha} \end{aligned}\right\} \quad (5)$$

Coefficients A_k, α are the numbers which are dependent on the interpolation formula chosen.

Substituting expressions (5) into the integral relations (4), we obtain a system of ordinary differential equations:

$$\frac{d}{dx} \delta \sum_{\alpha=0}^{\alpha=N} A_{k,\alpha} P_{i,\alpha} + Q_{i,\,k+1} - Q_{i,\,k} = \delta \sum_{\alpha=0}^{\alpha=N} A_{k,\alpha} F_{i,\alpha}$$

$$(i = 1, 2, \ldots, n; \qquad k = 0, 1, 2, \ldots, N - 1) \quad (6)$$

Together with boundary conditions (3) we shall have $n\,(N + 1)$ equations with regard to $n\,(N + 1)$ of unknown functions $u_{i,k}$ ($i = 1, 2, \ldots, n$; $k = 0, 1, 2, \ldots, N$), where $u_{i,k}$ is the value of function u_i at $y = y_k$.

The boundary conditions (2) will give those for the system of ordinary differential equations (6) + (3).

The solution of the system of equations (6) + (3) obtained when the region is divided into N bands will be called by us an N-approximation.

2. Now we consider the class of rather frequent problems in which the region where a solution is sought is indefinite.

Let the region, in which a solution of system (1) is to be found, be limited by the straight lines $x = a$, $x = b$, $y = 0$ (a can be $-\infty$, b can be $+\infty$) and by a curve $y = \delta(x)$ which itself is to be determined. In this case the system of boundary conditions (3) should be "over-determined", i.e. contain one additional condition.

$$\begin{aligned} &\text{at } y = 0 \qquad \psi_\nu (x, u_1, \ldots, u_n\, \delta) = 0 \quad (\nu = 1, 2, \ldots, 1) \\ &\text{at } y = \delta(x) \quad \psi_\nu (x, u_1, \ldots, u_n\, \delta) = 0 \quad (\nu = 1 + 1, 1 + 2, \ldots, n + 1) \end{aligned} \quad (3a)$$

(Some of the conditions (3a) can also be differential.)

Then the process of composing the approximating systems is analogous to the first case.

Now we divide the region into N curvilinear bands by lines

$$y = y_k = \frac{k}{N} \delta(x)$$

and integrate system (1) by y across each band. We obtain $n.N$ of integral relations

$$\frac{d}{dx}\int_{y_k}^{y_{k+1}} P_i dy - \left(\frac{k+1}{N} P_{i,\,k+1} - \frac{k}{N} P_{i,\,k}\right) \delta'(x) + Q_{i,\,k\mp 1}\, Q_{i,\,k} = \int_{y_k}^{y_{k+1}} F_i\, dy \tag{7}$$

$$(i = 1, 2, \ldots, n\,; \qquad k = 0, 1, 2, \ldots, N-1)$$

Applying interpolation formulas to the functions P_i and F_i, we shall have for the integrals the expressions (5) which, being substituted into the integral relations (7), give an N-approximation system.

$$\frac{d}{dx}(\delta \sum_{\alpha=0}^{\alpha=N} A_{k,\alpha} P_{i,\alpha}) - \left(\frac{k+1}{N} P_{i,\,k+1} - \frac{k}{N} P_{i,\,k}\right) \delta' + Q_{i,\,k+1} - Q_{i,\,k}$$

$$= \delta \sum_{\alpha=0}^{k=N} A_{k,\alpha} F_{i,\alpha} \tag{8}$$

$$(i = 1, 2, \ldots, n; \qquad k = 0, 1, \ldots, N-1)$$

System (8), together with boundary conditions (3a), gives $n(N+1)+1$ equations with regard to $n(N+1)+1$ sought functions $u_{i,\,k}$ ($i = 1, 2, \ldots, n$; $k = 0, 1, 2, \ldots, N$) and $\delta(x)$.

3. The method analogous to the one just described can also be applied in the case when the region is not limited by y (e.g. $d = \infty$).

In such cases the sought functions u_k are usually imposed by certain conditions of attenuation, limitation or gradual convergence to given values. Then in composition of approximating systems we can introduce artificial thickness of disturbance region $\delta(x)$ extending on it the conditions from infinity. If, for example, functions u_n (all or part of them) are converging to zero or to constant values, these conditions should be transferred to the line $y = \delta(x)$ and, if necessary, we add certain additional conditions, say, the condition of gradual convergence $\partial u_i/\partial y = 0$ at $y = \delta$) so as to obtain a complete system of boundary conditions.

Thickness of disturbance region $\delta(x)$ having been introduced, the solution process goes on as in the previous case. Of course, when the number of approximation N is increased, $\delta(x)$ will now not be converging to a definite function, but will grow unlimitedly.

Concluding the general description of the method, I would like to note that this method should not be regarded as a universal prescription. The task of a practician is to obtain a rather accurate solution of a concrete problem with a minimum investment. If we have chosen the method described for the solution the problem must be prepared in such a way that the method gives a good result already with a small approximation number, of N. But how to do this is already a question of the mathematician's art. I think that no universal prescriptions at all can exist here.

As for all other methods, the successful transformation of the initial system of equations, and the choice of a coordinate system and of a system of sought functions is of great importance. The choice of interpolation formulas for the transition from integral relations to the system of ordinary differential equations is essential in practice.

As to the convergence of the method in a usual meaning of the word, it can be, for the time being, successfully proved only for the simplest linear cases (as it also takes place for the method of straight lines). In practice we applied this method to the solution of nonlinear systems of partial differential equations and obtained the judgment on the convergence only by means of comparing the results of the calculations for successive number of approximation.

I shall illustrate it below on the examples of the solution of concrete problems.

Finally I note that the method of reducing the partial differential equations to systems of ordinary differential equations is convenient for the application on electronic computers because comparatively it little overloads the computer's storage unit.

Example

Let us consider the solution of a system

$$\begin{aligned} \frac{\partial u}{\partial y} - \frac{\partial v}{\partial x} &= 0 \\ \frac{\partial}{\partial x}(1 - x)u + \frac{\partial v}{\partial y} &= 0 \end{aligned} \tag{9}$$

within a semi-band

$$x \geqslant 0, \qquad 0 \leqslant y \leqslant 1.$$

This system represents an example of a mixed system: when $x < 1$, the system is elliptical, and, when $x > 1$, the system is hyperbolical. It can be considered as the simplest mathematical model for the problem of flow with detached shock-wave. Giving functions u or v on the boundary of the region in the elliptical part determines a continuous solution over the whole elliptical part of the region and in the hyperbolic part up to the characteristics proceeding from points (1, 0) and (1, 1) (see Fig. 1). At the same time giving functions u or v on the boundary of the band when $x > 1$ in no way influences the solution in the above determined part of the region (ABCDE in Fig. 1). Let us see how this property manifests itself in the solution by method of integral relations.

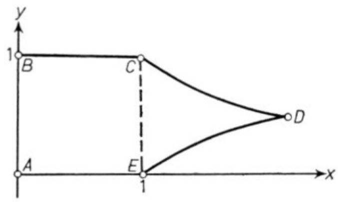

FIG. 1

For complete clearness we take concrete boundary conditions.

$$\begin{aligned} &\text{at } x = 0 \qquad && u = 0 \\ &\text{at } y = 0 && v = 0 \\ &\text{at } y = 1 && u = x \end{aligned} \tag{10}$$

The integral relations used for the construction of a system of ordinary differential equations, for the first approximation, can be written as follows

$$\begin{aligned} &\frac{\mathrm{d}}{\mathrm{d}x}\int_0^1 v\,\mathrm{d}y - u_1 + u_0 = 0 \\ &\frac{\mathrm{d}}{\mathrm{d}x}(1-x)\int_0^1 u\,\mathrm{d}y + v_1 = 0 \end{aligned} \tag{11}$$

where u_0 is the value of u at $y = 0$
u_1 is the value of u at $y = 1$
v_1 is the value of v at $y = 1$
$v_0 = 0$ is the value of v at $y = 0$.

Using ordinary interpolation in the form of polynoms, we obtain for the first approximation

$$u = u_0 + (u_1 - u_0)y \qquad v = v_1 . y$$

and the system of ordinary differential equations will assume the following form

$$\frac{\mathrm{d}v_1}{\mathrm{d}x} - 2u_1 + 2u_0 = 0$$

$$\frac{\mathrm{d}}{\mathrm{d}x}(1-x)(u_0 + u_1) + 2v_1 = 0$$

or, since, according to (10) $u_1 = x$

$$\begin{aligned} &\frac{\mathrm{d}v_1}{\mathrm{d}x} + 2u_0 = 2x \\ &\frac{\mathrm{d}}{\mathrm{d}x}(1-x)\,u_0 + 2v_1 = 2x - 1 \end{aligned} \tag{12}$$

The boundary conditions (10) give only one condition, $u_0(0) = 0$.

But the point $x = 1$ is a point of singularity of the system (12). The requirement of continuity of the solution in the transition through the transition line ($x = 1$) imposes an additional condition for the uniquity of the solution of system (12). This continuous solution has obviously the form

$$u_0 = x - 1 + \frac{1}{\sqrt{(1-x)}} \cdot \frac{I_1[4\sqrt{(1-x)}]}{I_1(4)}$$
$$v_1 = 2x - \frac{3}{2} + \frac{I_0[4\sqrt{(1-x)}]}{I_1(4)} \qquad (13)$$

The solution in any approximation is constructed analogously.

This example has been given in order to illustrate the specific features of the solution of mixed problems. It is characteristic of them that the boundary conditions in the hyperbolic part of the region do not influence the solution in the so-called region of influence (in our example, the region ABCDE). In approximate treatment this fact is manifested in that, for the system of ordinary differential equations, no boundary conditions are available at the right hand (on the side of the hyperbolic region). The uniqueness of the solution is provided by the demand of its continuity in the transition through points of singularity.

RESULTS OF THE SOLUTION OF SOME AERODYNAMICAL PROBLEMS

The method described has been applied to the solution of a number of problems of high-speed aerodynamics.

I shall cite the results of the calculations of three problems:

1. Subsonic flow past ellipses and ellipsoids. This work has been carried out by P. I. Chushkin, a scientific worker of the Computing Center, The Academy of Sciences of the U.S.S.R.

2. Flow past ellipses with a sonic velocity at the infinity. This work has also been carried out by P. I. Chushkin.

3. Supersonic flow past a circular cylinder. This work has been carried out by O. M. Bielotserkovsky, a scientific worker of the Computing Center.

In the first case the equations are written in elliptical coordinates. It is enough to find the solution in one quadrant. In elliptical coordinates the region is transformed into a semi-band

$$\xi_0 \leqslant \zeta \leqslant \infty, \qquad 0 < \eta < \frac{\pi}{2}$$

Interpolation formulas are taken in the form of trigonometrical polynoms. The calculation results are shown in Figs. 2, 3.

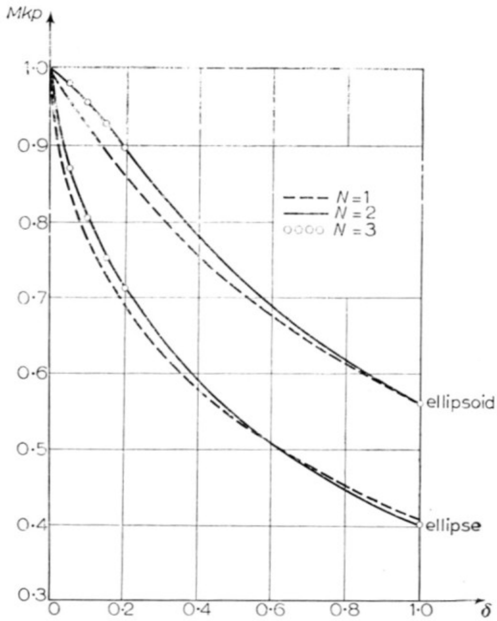

FIG. 2. The values of the critical Mach number for ellipses and ellipsoids *versus* relative thickness δ with different approximations.

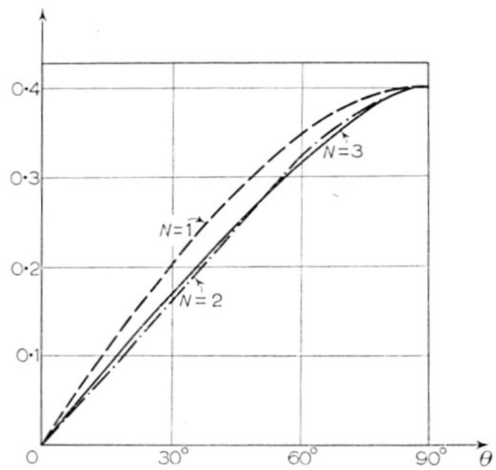

FIG. 3. Velocity distribution past circle at critical Mach number for different approximations.

In the second case the region in which the solution is sought is limited by the abscissa axis, by ellipse surface and by the unknown limiting characteristics of the first family going to infinity. The calculation results are shown in Figs. 4, 5, and 6.

In the third case the region in which the solution is found is limited by the shock wave (whose shape is unknown), by the section of abscissa axis and by the surface of the circular cylinder. Since the problem is mixed, the fourth boundary is absent and the uniqueness of the solution is provided by the demand of continuity of solution in the transition from the elliptical into hyperbolical part of the region in the same manner as in the example we have considered. The calculation results are shown in Figs. 7, 8, 9, 10, and 11.

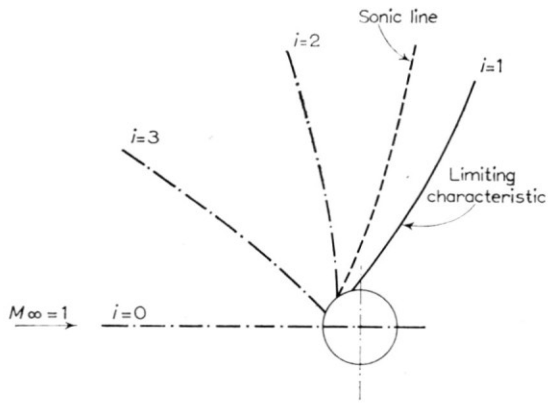

FIG. 4. The scheme of composition of integral relations for calculation of flow past circle at free-stream Mach number $M_{\infty} = 1$.

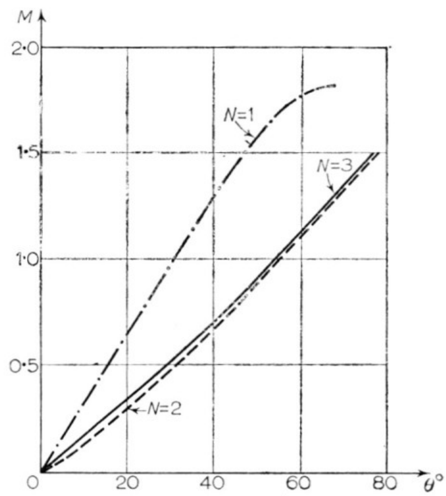

FIG. 5. Mach number distribution past circle at free-stream Mach number $M_{\infty} = 1$.

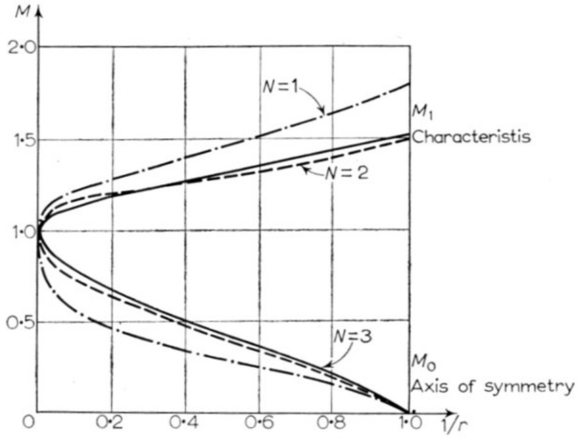

FIG. 6. Mach number distribution along axis of symmetry and limiting characteristic for circle at free-stream Mach number $M_\infty = 1$ for different approximations.

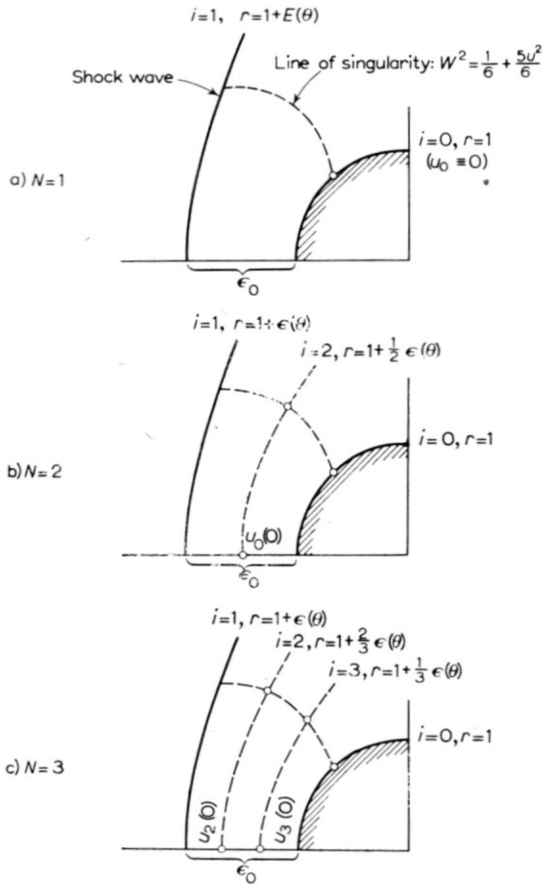

FIG. 7. The scheme of composition of integral relations for calculation of supersonic flow past circle with detached shock wave.

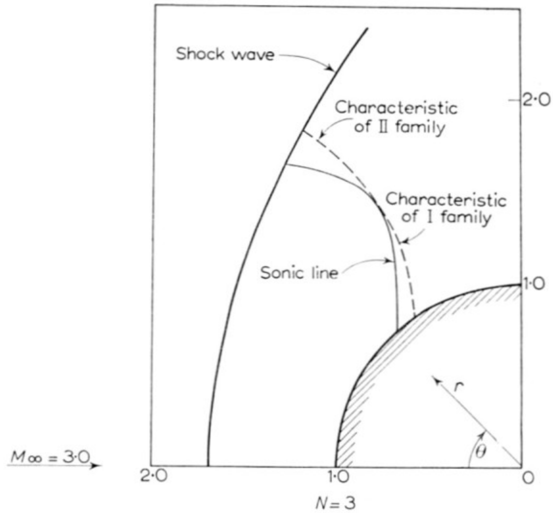

FIG. 8. The flow past circle at free-stream Mach number $M_\infty = 3$ for $3 - d$ approximation.

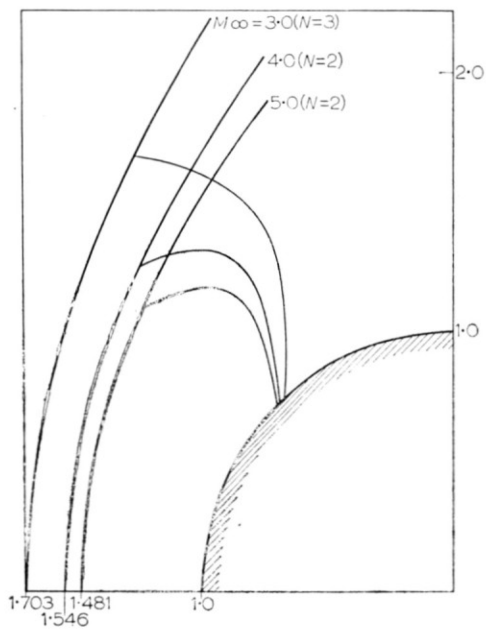

FIG. 9. Shock waves and sonic lines for circle at different free-stream Mach numbers.

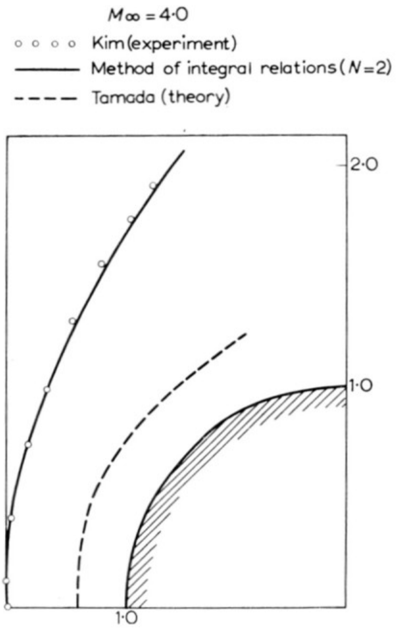

FIG. 10. Shock wave for circle at free-stream Mach number $M_\infty = 4$ as compared with the experiment.

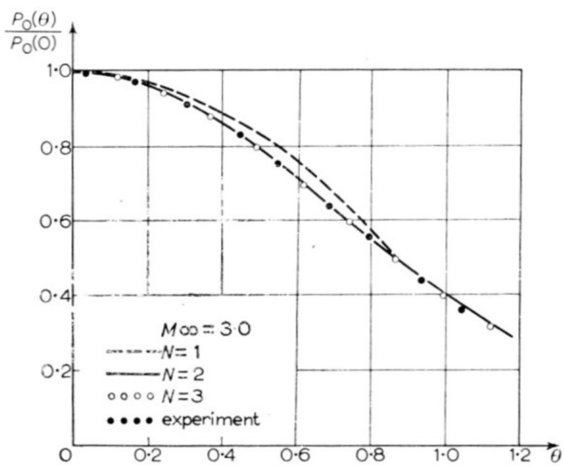

FIG. 11. Pressure distribution past circle at free-stream Mach number $M_\infty = 3$ as compared with the experiment.

DISCUSSION

H. H. PEARCEY*: For the case of the circular cylinder with detached shock wave, the agreement that Mr. Dorodnicyn showed us between his theoretical results and experimental ones was most impressive. We shall be most interested to know whether similarly good agreement is obtained for a two-dimensional aerofoil at a free-stream Mach number of 1·0, and, if so, whether this holds for a lifting aerofoil. Also, I look forward to comparing his results for aerofoils with ones that would be given for the same shapes by a semi-empirical method developed by C. S. Sinnott at the N.P.L. and shortly to be published in the *Journal of the Aeronautical Sciences*. We adopted this semi-empirical approach because hitherto we have not found any purely theoretical method that will give physically realistic results and good comparison with experiment. If, as seems likely, the method now presented to us will do this, then it should be of great value in producing results for specific shapes and, probably, more important still, in elucidating the essential mechanism of these flows.

In our work on two-dimensional aerofoils, aimed at deriving section shapes for improved performance and delayed boundary-layer separation on sweptback wings, we attached considerable importance to the surface pressure distribution at M = 1·0, because for each particular shape it characterises the distribution that will be obtained in the locally supersonic flow upstream of the shock wave for lower free-stream Mach numbers.

Sinnott, in an extension of his work mentioned above, has used this fact in the prediction of pressure distributions for lifting aerofoils at Mach numbers below 1·0, when the shock wave impinges on the surface between the leading and trailing edges, instead of at the trailing edge as at $M = 1{\cdot}0$. Again in this case, Sinnott's method predicts results that are so far the only ones that we have found to be physically realistic and to agree well with experiment. Can Mr. Dorodnicyn say whether his method can be used for such cases? If so, then again it should be of great value, in this instance for the derivation of optimum section shapes for sweptback wings on which both leading and trailing edges are behaving subsonically.

A. A. DORODNICYN: 1. The mathematical problem discussed in my report is the solution of differential equations of aerodynamics for free-stream Mach number = 1 in the domain bounded by the first characteristics going to infinity.

For the complete solution of the problem of flow about an aerofoil the usual method of characteristics must be used behind the mentioned characteristics.

The presence of shock waves between the leading and trailing edges makes impossible the use of the method of characteristics.

2. The author of the work on flow about a profile by Mach number 1 (P. I. Chushkin) had no experimental result in his disposition exactly corresponding to the calculated cases.

However, taking into account the known experimental result that in the vicinity of Mach number 1 the distribution of local Mach numbers depends slightly on free-stream Mach number, one can conclude that calculations are in good coincidence with experiment.

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